Exercise 12

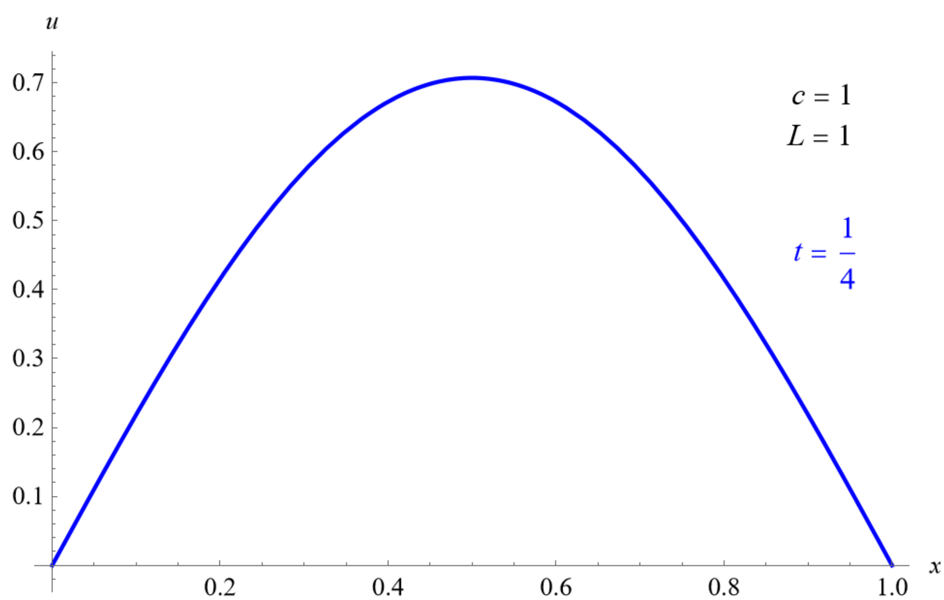
In the second frame in Figure 5 (for $t = 1/4$), the maximum of the displacement appears to be larger than $1/2$. Explain this, and if possible compute the value of the maximum.

Solution

Figure 5 shows the graph of

$$u(x,t) = \sin\frac{\pi x}{L}\sin\frac{\pi ct}{L}, \quad 0 < x < L,$$

a solution to the wave equation on the interval $0 < x < L$ with fixed ends, versus x at several times. The graph at $t = 1/4$ for $c = 1$ and $L = 1$ is shown below.



To find the maximum of this curve shown, differentiate u with respect to x

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}\left(\sin\frac{\pi x}{L}\sin\frac{\pi ct}{L}\right)\\ &= \frac{\pi}{L}\cos\frac{\pi x}{L}\sin\frac{\pi ct}{L},\end{aligned}$$

plug in $t = 1/4$ and $c = 1$ and $L = 1$,

$$\frac{\partial u}{\partial x}\left(x, \frac{1}{4}\right) = \pi\cos\pi x\sin\frac{\pi}{4},$$

and set it equal to zero.

$$\pi\cos\pi x\sin\frac{\pi}{4} = 0$$

Solve for x.

$$\cos\pi x = 0$$

$$\pi x = \frac{1}{2}(2n-1)\pi, \quad n = 1, 2, 3, \ldots$$

At the first turning point in the graph $n = 1$.

$$\pi x = \frac{\pi}{2}$$

Consequently, the x-coordinate of the maximum is

$$x = \frac{1}{2}.$$

To get the string's amplitude, plug this x-coordinate and $t = 1/4$ into the formula for u with $c = 1$ and $L = 1$.

$$u\left(\frac{1}{2}, \frac{1}{4}\right) = \sin\frac{\pi}{2}\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \approx 0.707$$

The maximum occurs at $x = 1/2$, but the value of the maximum is $1/\sqrt{2}$.